Entropy of Random Permutation Set

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Abstract

Recently, a new kind of set, named as Random Permutation Set (RPS), is proposed. RPS takes the permutation of a certain set into consideration, which can be regarded as a generalization of evidence theory. Uncertainty is an important feature of RPS. A straightforward question is how to measure the uncertainty of RPS. To address this problem, the entropy of RPS is presented in this paper. When the order of elements in permutation event is ignored, the proposed RPS entropy degenerates into Deng entropy in evidence theory. When each permutation event is limited to containing just one element, the proposed RPS entropy degenerates into Shannon entropy in probability theory. Hence the RPS entropy can be regarded as the generalization of Deng entropy and Shannon entropy. Numerical examples are illustrated the efficiency of the proposed RPS entropy.

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1. Introduction

Uncertainty plays an important role in our daily life, different theories have been developed to deal with uncertainty [60, 28, 47, 40, 34], such as probability theory [19], rough sets [29, 22, 8], fuzzy sets [53, 30], type-2 fuzzy sets [31, 3], Evidence theory [39, 51], Z-numbers [54, 18], D number [10], belief rule [4], three-way decision [52, 58], etc. These methods are applied in various areas, like traffic control [21], decision-making [14, 57, 27], classification [26, 25], risk assessment [38], and so on [15, 7, 20].

Most existing theories are based on set theory. Set theory provides a tool for describing the collection of objects or elements, which is important and fundamental in math science [17]. For instance, the sample space of an experiment in probability theory is the set that contains all possible outcomes of that experiment [19]. In evidence theory, the power set is the set that considers all possible subsets of frame of discernment, which is a basis for basic probability assignment (BPA) [9, 32].

Recently, for exploring the meaning of the power set in evidence theory, a possible explanation of power set is proposed from the view of Pascal's triangle and combinatorial number [36]. Then inspired by the idea of replacing combinatorial number with permutation number, Deng proposed a new type of set, called Random permutation set (RPS) [13]. RPS

consists of permutation event space (PES) and permutation mass function (PMF). The PES of a certain set considers all the permutation of that set. The elements of PES are called permutation events. PMF describes the chance of a certain permutation event that would happen. RPS is compatible with evidence theory and probability theory [13].

RPS provides a new perspective to deal with uncertainty regarding the order of elements, a straightforward question is that how to measure the uncertainty of RPS. Reviewing the existing uncertainty measures, various kinds of entropy functions have been presented, such as Shannon entropy in probability theory [33], Deng entropy in evidence theory [11], and fuzzy entropy in fuzzy sets [37]. In this paper, the entropy of RPS is presented for handling the uncertainty measure. The entropy of RPS is compatible with Deng entropy and Shannon entropy. When the order of elements in permutation events is ignored, the proposed RPS entropy will degenerate into Deng entropy. When each permutation event is limited to containing just one element, the proposed RPS entropy will degenerate into Shannon entropy. Several numeric examples are illustrated the efficiency of the proposed RPS entropy.

The rest of this article is as follows. Section 2 introduces the preliminaries. Section 3 presents the entropy of RPS. Section 4 uses some numerical example to illustrate the presented RPS entropy. Section 5 makes a brief conclusion.

2. Preliminaries

In this section, we briefly review some preliminaries of this paper.

2.1. Random permutation set

Random permutation set (RPS) is a novel set consisting of permutation event space (PES) and permutation mass function (PMF) [13]. Some basic definitions of RPS are given as follows.

Definition 2.1 (Permutation event space). *Given a fixed set of N mutually exclusive and exhaustive elements $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, its permutation event space (PES) is a set of all possible permutations of Θ defined as follows:*

$$
\begin{aligned}
PES\,(\Theta) = & \{\, A_{ij} \,|\, i = 0, ..., N;\, j = 1, ..., P(N, i) \,\} \quad (1) \\
= & \{\, \emptyset, (\theta_1), (\theta_2), ..., (\theta_N), (\theta_1, \theta_2), (\theta_2, \theta_1), ..., (\theta_{N-1}, \theta_N), \\
& (\theta_N, \theta_{N-1}), ..., (\theta_1, \theta_2, ..., \theta_N), ..., (\theta_N, \theta_{N-1}, ..., \theta_1) \,\} \quad (2)
\end{aligned}
$$

in which $P(N, i)$ is the i-permutation of N defined as $P(N, i) = \frac{N!}{(N-i)!}$. The element A_{ij} in PES is called the permutation event, which is a tuple representing a possible permutation of Θ, where i indicates the index for the cardinality of A_{ij} and j denotes the index for the possible permutation.

Definition 2.2 (Random permutation set). *Given a fixed set of N mutually exclusive and exhaustive elements $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, its random permutation set (RPS) is a set of pairs defined as follows:*

$$
RPS\,(\Theta) = \{\langle A, \mathscr{M}\,(A)\rangle \mid A \in PES\,(\Theta)\} \quad (3)
$$

where $\mathscr{M}$ is called the permutation mass function (PMF), which is defined as:

$$
\mathscr{M} : PES(\Theta) \to [0, 1] \quad (4)
$$

constrained by $\mathscr{M}\,(\emptyset) = 0$ and $\sum_{A \in PES(\Theta)} \mathscr{M}\,(A) = 1$.

Some important properties about RPS are discussed in [13]. Specifically, the PES of RPS is compatible with the power set in evidence theory and sample space in probability theory. The PMF of RPS is compatible with BPA and probability distribution.

2.2. Evidence theory

Evidence theory (Dempster-Shafer evidence theory) provides an efficient tool for dealing with uncertainty [12, 61, 35]. Researchers have developed evidence theory theoretically, especially for some open issues [59, 48]. For instance, the decision-making model based on mass function [5], uncertainty measure of mass function [49] and the complex evidence theory [43, 46, 45]. With the superiority in uncertain environment, evidence theory also has been widely applied in engineering fields [16, 6].

Some basic conceptions about evidence theory are summarized as follows [9, 32].

Definition 2.3 (Frame of discernment). *Let Ω, called the frame of discernment (FOD), denotes a fixed set of N mutually exclusive and exhaustive elements, indicated by*

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \tag{5}$$

$PS(\Omega)$ is the power set of Ω, which contains all subsets of Ω and has 2^N elements, indicated by

$$\begin{aligned} PS(\Omega) &= \{A_1, A_2, \cdots, A_{2^N}\} \\ &= \{\varnothing, \{\omega_1\}, \{\omega_2\}, \cdots, \{\omega_N\}, \\ &\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \cdots, \{\omega_1, \omega_N\}, \cdots, \Omega\} \end{aligned} \tag{6}$$

Definition 2.4 (Mass function). *Given a FOD of Ω, a basic probability assignment (BPA), also called a mass function, is a mapping from $PS(\Omega)$ to [0, 1], formally defined by:*

$$m : PS(\Omega) \rightarrow [0, 1] \tag{7}$$

constrained by

$$m(\varnothing) = 0 \, and \sum_{A \in 2^\Omega} m(A) = 1 \tag{8}$$

A is called a focal element if $m(A) > 0$.

2.3. Shannon entropy

Shannon entropy, which was proposed by Shannon, addressed the problem of quantitative measurement of information [33]. The related definition is described below:

Definition 2.5. *Given a probability distribution in a certain state space, Shannon entropy is defined as*

$$H_{SE}(p) = -\sum_{i=1}^{n} p_i log_b p_i \tag{9}$$

where n is the number of basic states, b is the base of logarithm. p_i denotes the probability of state i appears, $\sum_{i=1}^{n} p_i = 1$.

2.4. Deng entropy

Entropy plays a significant role in uncertainty modelling [50, 24, 55, 56, 1], especially in complex systems [44, 2]. Deng entropy [11] is a kind of belief entropy, which measures the uncertainty in evidence theory and applied in many fields [42, 23, 41]. An important feature of Deng entropy is that it is compatible with Shannon entropy.

Definition 2.6 (Deng entropy). *Given a mass function distribution defined on Ω, Deng entropy is defined as [11]:*

$$H_{DE}(m) = -\sum_{A \in PS(\Omega)} m(A) \log \frac{m(A)}{2^{|A|} - 1} \tag{10}$$

3. Entropy of random permutation set

For measuring the uncertainty of RPS, the entropy of RPS is presented in this section.

Definition 3.1 (Entropy of random permutation set). *Given a RPS denoted as $RPS(\Theta) = \{\langle A_{ij}, \mathscr{M}(A_{ij})\rangle \mid A_{ij} \in PES(\Theta)\}$, the entropy of RPS is defined as:*

$$H_{RPS}(\mathscr{M}) = -\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} \mathscr{M}(A_{ij}) \log\left(\frac{\mathscr{M}(A_{ij})}{F(i)-1}\right) \tag{11}$$

where $P(N,i) = \frac{N!}{(N-i)!}$ is the i-permutation of N and $F(i) = \sum_{k=0}^{i} P(i,k) = \sum_{k=0}^{i} \frac{i!}{(i-k)!}$ is the sum from 0-permutation of i to i-permutation of i.

Proposition 3.1. *If the order of elements in permutation event is ignored, the proposed RPS entropy will degenerate into Deng entropy in evidence theory.*

Proof 3.1. *If the order of elements in permutation event is ignored, the permutation number $P(N,i)$ degenerates into combinatorial number $C(N,i)$, and $F(i)$ should be calculated as $F(i) = \sum_{k=0}^{i} C(i,k) = 2^i$.*

$$\begin{aligned} H_{RPS}(\mathscr{M}) &= -\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} \mathscr{M}(A_{ij}) \log\left(\frac{\mathscr{M}(A_{ij})}{F(i)-1}\right) \\ &= -\sum_{i=1}^{N}\sum_{j=1}^{C(N,i)} \mathscr{M}(A_{ij}) \log\left(\frac{\mathscr{M}(A_{ij})}{2^i-1}\right) \\ &= -\sum_{i=1}^{2^N} \mathscr{M}(A_i) \log\left(\frac{\mathscr{M}(A_i)}{2^{|A_i|}-1}\right) \end{aligned} \tag{12}$$

As discussed in Ref. [13], under the situation of ignoring the order of elements in permutation event, the PES of RPS degenerates to the power set, and the PMF of RPS degenerates to BPA in evidence theory. Hence Eq. (12) is also calculated as

$$\begin{aligned} H_{RPS}(\mathscr{M}) &= -\sum_{A_i \in PS(\Omega)} m(A_i) \log\left(\frac{m(A_i)}{2^{|A_i|}-1}\right) \\ &= H_{DE}(m) \end{aligned} \tag{13}$$

Therefore, when the order of elements in permutation event is ignored, the proposed RPS entropy degenerates to Deng entropy.

Proposition 3.2. *When each permutation event is limited to containing just one element, the proposed RPS entropy will degenerate into Shannon entropy in probability theory.*

Proof 3.2. *If each permutation event is limited to containing just one element,* $i = 1$, $F(i) = F(1) = 2$.

$$\begin{aligned} H_{RPS}(\mathscr{M}) &= -\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} \mathscr{M}\left(A_{ij}\right)\log\left(\frac{\mathscr{M}\left(A_{ij}\right)}{F\left(i\right)-1}\right) \\ &= -\sum_{j=1}^{P(N,1)} \mathscr{M}\left(A_{1j}\right)\log\left(\frac{\mathscr{M}\left(A_{1j}\right)}{F(1)-1}\right) \\ &= -\sum_{j=1}^{N} \mathscr{M}\left(A_{1j}\right)\log\left(\mathscr{M}\left(A_{1j}\right)\right) \end{aligned} \tag{14}$$

As discussed in Ref. [13], under the situation of containing just one element in each permutation event, the PES of RPS degenerates to the sample space, and the PMF of RPS degenerates to probability distribution. Hence Eq. (14) is also calculated as

$$\begin{aligned} H_{RPS}(\mathscr{M}) &= -\sum_{j=1}^{N} p_j \log p_j \\ &= H_{SE}(p) \end{aligned} \tag{15}$$

Therefore, when each permutation event is limited to containing just one element, the proposed RPS entropy degenerates to Shannon entropy.

In conclusion, from Propositions 3.1 and 3.2, we can conclude that the proposed RPS entropy is compatible with Deng entropy and Shannon entropy, as shown in Fig. 1.

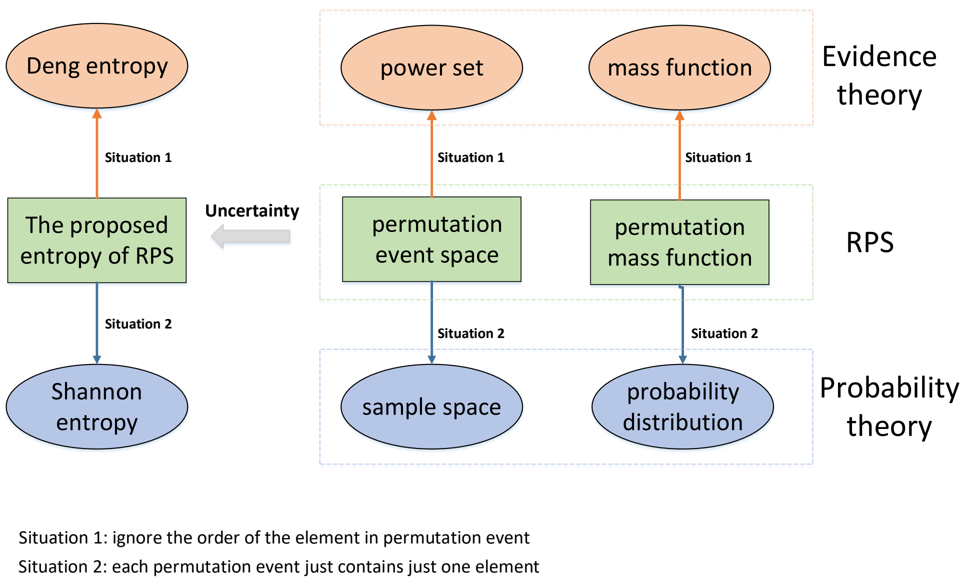

Figure 1: The relationship between the proposed RPS entropy, Deng entropy and Shannon entropy

4. Numerical examples and discussions

In this section, some numerical examples are shown to illustrate the presented entropy of RPS.

Example 4.1. *Given the fixed set of* $\Theta = \{X, Y, Z\}$, *a RPS defined on* Θ *is given as follows:*

$$RPS_1(\Theta) = \{< (X), 0.4 >, < (Y, Z), 0.1 >, < (X, Y, Z), 0.15 >, < (Y, Z, X), 0.35 >\} \tag{16}$$

Based on Eq. (11), the associated entropy of RPS_1 *is calculated as*

$$\begin{aligned} H_{RPS_1}(\mathscr{M}) &= -\sum_{i=1}^{3}\sum_{j=1}^{P(3,i)} \mathscr{M}(A_{ij}) \log\left(\frac{\mathscr{M}(A_{ij})}{F(i)-1}\right) \\ &= -0.4 * log\left(\frac{0.4}{F(1)-1}\right) - 0.1 * log\left(\frac{0.1}{F(2)-1}\right) \\ &- 0.15 * log\left(\frac{0.15}{F(3)-1}\right) - 0.35 * log\left(\frac{0.35}{F(3)-1}\right) \\ &= -0.4 * log\left(\frac{0.4}{2-1}\right) - 0.1 * log\left(\frac{0.1}{5-1}\right) \\ &- 0.15 * log\left(\frac{0.15}{16-1}\right) - 0.35 * log\left(\frac{0.35}{16-1}\right) \\ &= 2.8975 \end{aligned} \tag{17}$$

Followed by Example 4.1, consider the following two scenarios:

- If the order of elements in permutation events is ignored, (X, Y, Z) and (Y, Z, X) are the same sets in Eq. (16), the RPS_1 is updated as

$$RPS_2(\Theta) = \{< (X), 0.4 >, < (Y, Z), 0.1 >, < (X, Y, Z), 0.5 >\} \tag{18}$$

The associated RPS_2 entropy is calculated as

$$\begin{aligned} H_{RPS_2}(\mathscr{M}) &= -0.4 * log\left(\frac{0.4}{F(1)-1}\right) - 0.1 * log\left(\frac{0.1}{F(2)-1}\right) - 0.5 * log\left(\frac{0.5}{F(3)-1}\right) \\ &= -0.4 * log\left(\frac{0.4}{2-1}\right) - 0.1 * log\left(\frac{0.1}{3-1}\right) - 0.5 * log\left(\frac{0.5}{7-1}\right) \\ &= 1.8656 \end{aligned} \tag{19}$$

Actually, when the order of elements in permutation events is ignored, the permutation event space degenerates into the power set in evidence theory, i.e., $PES(\Theta)$ is the same as $PS(\Omega)$. Therefore, the RPS_2 in Eq. (18) can be regarded a mass function in evidence theory, just as

$$m(\{X\}) = 0.4, m(\{Y, Z\}) = 0.1, m(\{X, Y, Z\}) = 0.5 \tag{20}$$

Based on Eq. (10), the associated Deng entropy is calculated as

$$\begin{aligned} H_{DE}(m) &= -0.4 * log\left(\frac{0.4}{2^1-1}\right) - 0.1 * log\left(\frac{0.1}{2^2-1}\right) - 0.5 * log\left(\frac{0.5}{2^3-1}\right) \\ &= 1.8656 \end{aligned} \tag{21}$$

which is the same as H_{RPS_2} in Eq. (19). As a result, if the order of elements in PES is ignored, the proposed RPS entropy will degenerate into Deng entropy in evidence theory.

- For the RPS_2 in Eq. (16), let $A_1 = (X)$, $A_2 = (Y, Z)$, $A_3 = (X, Y, Z)$. If these permutation events just contain one element, and they are mutually exclusive and independent, just as

$$RPS_3(\Theta) = \{< (A_1), 0.4 >, < (A_2), 0.1 >, < (A_3), 0.5 >\} \tag{22}$$

The associated entropy of RPS_3 is calculated as

$$
\begin{aligned}
& H_{RPS_3}(\mathscr{M}) = 0.4 * log\left(\frac{0.4}{F(1)-1}\right) + 0.1 * log\left(\frac{0.1}{F(1)-1}\right) + 0.5 * log\left(\frac{0.5}{F(1)-1}\right) \\
& = -(0.4 * log\left(\frac{0.4}{2-1}\right) + 0.1 * log\left(\frac{0.1}{2-1}\right) + 0.5 * log\left(\frac{0.5}{2-1}\right)) \\
& = 1.3610
\end{aligned}
\tag{23}
$$

Actually, when each permutation event just contains one element and they are mutually exclusive and independent, the permutation event space is equivalent to the sample space composed of basic events in probability theory. Therefore, the RPS_3 in Eq. (22) can be regarded a probability distribution, just as

$$p(X) = 0.4, p(Y) = 0.1, p(Z) = 0.5 \tag{24}$$

Based on Eq. (9), the Shannon entropy is calculated as

$$
\begin{aligned}
H_{SE}(p) &= -0.4 * log\,(0.4) - 0.1 * log\,(0.1) - 0.5 * log\,(0.5) \\
&= 1.3610
\end{aligned}
\tag{25}
$$

which is the same as H_{RPS_3} in Eq. (23). As a result, if each permutation event just contains one element and they are mutually exclusive and independent, the proposed RPS entropy will degenerate into Shannon entropy.

Based on the discussion of these two scenarios, it can be concluded that the proposed RPS entropy is compatible with Deng entropy and Shannon entropy, as shown in Fig. 2.

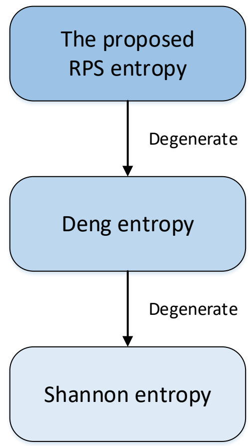

Figure 2: the proposed RPS entropy is compatible with Deng entropy and Shannon entropy

Example 4.2. *Given a fixed set of* $\Theta = \{1, 2, 3, \ldots, 10\}$*. A RPS defined on* Θ *is shown as follows.*

$$RPS(\Theta) = \{< (3, 4, 5), 0.4 >, < (6), 0.1 >, < (X), 0.8 >, < (1, 2, 3, \ldots, 10), 0.1 >\}$$

If the order in permutation events is ignored, the RPS is a mass function in evidence theory.

$$m(\{3, 4, 5\}) = 0.4, m(\{6\}) = 0.1, m(\{X\}) = 0.8, m(\Theta) = 0.1$$

When X changes from 1 to $1, 2, 3, \ldots, 10$, the values of Deng entropy and the RPS entropy are calculated, which are shown in Table 1 and Figure 3.

From Table 1 and Figure 3, the results shows that as the size of subset X rises, both Deng entropy and the proposed RPS entropy increase monotonously. It is reasonable that the entropy values increase when the uncertainty involving a set increases. Importantly, with respect to the

Table 1: Deng entropy and the proposed RPS entropy when X chenges

X	$Dengentropy$	$TheRPSentropy$
1	2.1622	3.5406
1,2	3.4301	5.1406
1,2,3	4.4080	6.6662
1,2,3,4	5.2877	8.3406
1,2,3,4,5	6.1255	10.2161
1,2,3,4,5,6	6.9440	12.2876
1,2,3,4,5,6,7	7.7531	14.5341
1,2,3,4,5,6,7,8	8.5576	16.9342
1,2,3,4,5,6,7,8,9	9.3599	19.4701
1,2,3,4,5,6,7,8,9,10	10.1610	22.1277

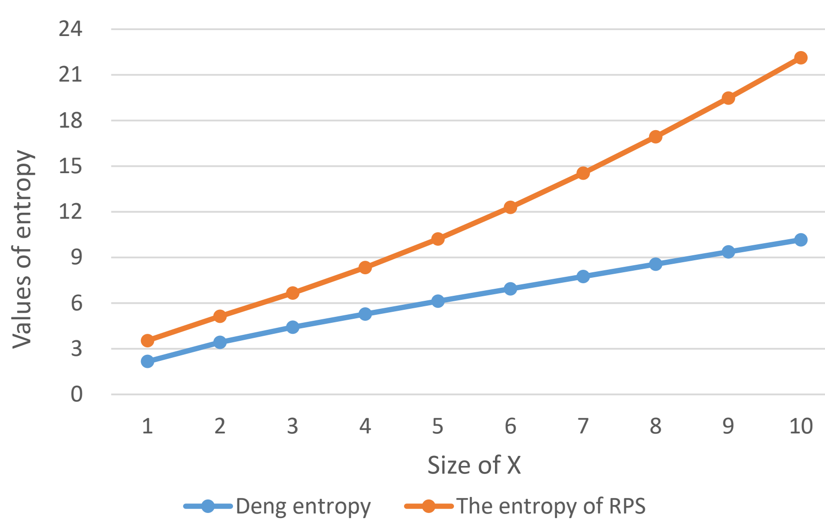

Figure 3: The trend of Deng entropy and the proposed RPS entropy when X changes

same size of X, the value of RPS entropy is larger than that of Deng entropy. This means that, the uncertainty of a RPS is larger than that of a mass function, since RPS takes permutation of a certain set into account while a mass function does not consider that.

5. Conclusion

Entropy is an important function to measure uncertainty. The main contribution of this paper is that the entropy of a new kind of set, named as Random permutation set (RPS), is first proposed. The proposed RPS entropy are compatible with Deng entropy and Shannon entropy. When the order of elements in permutation event is ignored, the proposed RPS entropy degenerates into Deng entropy. When each permutation event is limited to containing just one element, the proposed RPS entropy degenerates into Shannon entropy. Numerical examples are illustrated the proposed entropy of RPS. The new entropy provides a promising way for measuring the uncertain degree and handling ranked uncertain information.

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